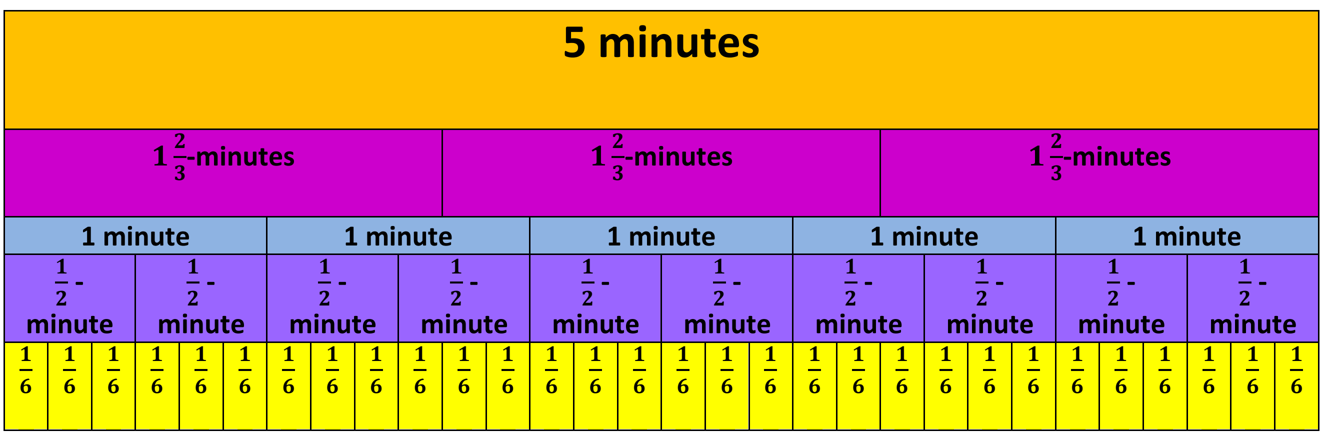
A Republic of Mathematics publication

DIVISION OF FRACTIONS

Gary E. Davis & Catherine A. Pearn



A guide for elementary and middle school teachers

DIVISION OF FRACTIONS

Version 1.0

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Contents

Division by whole numbers

Before looking at division of fractions let's look at division of one whole number by another.

Suppose we want to divide 20 by 5.

There are two ways to do this, and for young children these two ways rely on quite different abilities and skills.

In one of these ways we form equal sized groups and ask: "How many are there in each group?" The other way, we say how many are in each group and ask: "How many groups are there?"

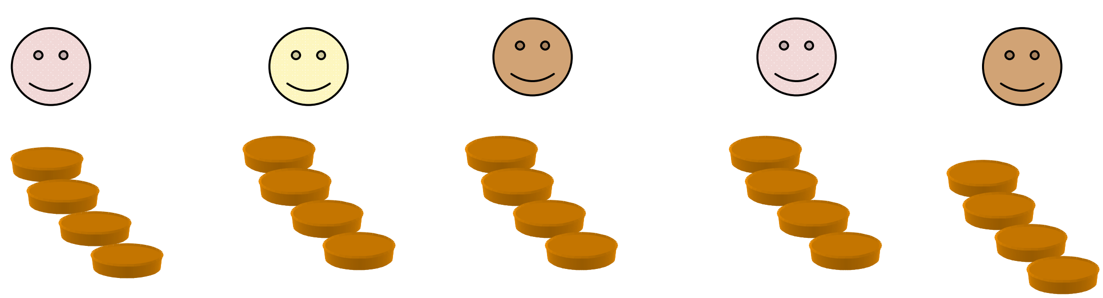
Partitive division

We can form 5 groups, of equal size, and find that there are 4 objects in each group.

The result of dividing 20 by 5 is the size of each group – namely 4 – and we write $20 \div 5 = 4$

This is called partitive division, because it focuses on partitioning, or splitting, the collection of objects into so many groups of equal size.

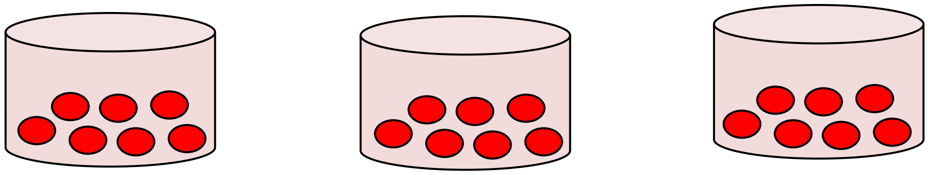
The first step in partitive division is very simple: it is just like sharing, or dealing out, that young children know how to do without being taught. For example we could say: "Here are 20 cookies. Can you share them all out to these 5 dolls?" Then, when children do that, we can ask: "How many did each doll get?" Now the children can count, and find that $20 \div 5 = 4$.



There are 5 dolls, and each doll gets 4 cookies

Notice that partitive division does not really answer a counting question – a "how many?" question –until the end, when we have split the objects into equal groups. At that point, children count, in a simple way, using one-to-one correspondence, how many cookies each doll has been dealt.

Another example might be partitioning 21 objects into 3 groups. We write this as a division problem as follows: $21 \div 3$. Here we want to know if we partition, or split, 21 objects into 3 groups, each of equal size, how many objects will be in each group?



When we ask children to carry out partitive division we are fixing a number of spots, or places, and asking them to share out objects equally to each of the spots. Then we ask: "How many objects are there at each spot?"

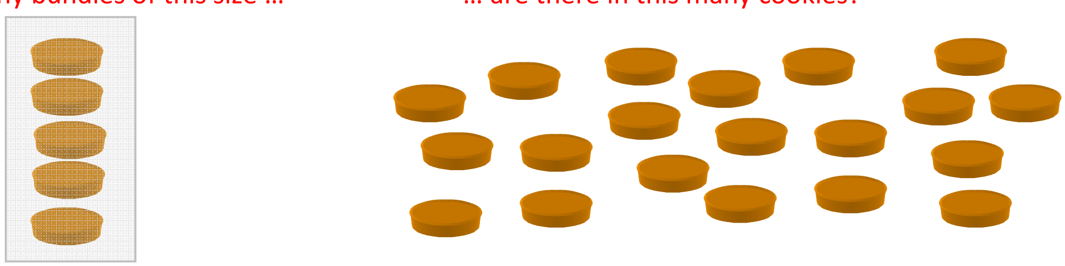
Notice how simple this process is: it relies on children's abilities to share out evenly, and to count using one-to-one correspondence. That's all. So partitive division is relatively simple for children in the earliest grades.

Quotitive division

There is another way to think about division, for example how many groups of size 5 can we form from 20 objects. We write this as a division problem as $20 \div 5 = 4$. This way is called quotitive division and involves forming groups of 5 objects and counting how many groups we get.

To carry out this process a child needs to bundle 5 objects into a group, and then keep bundling until there are no more groups. The number of bundles is the answer to the division question.

How many bundles of this size … … are there in this many cookies?



his is conceptually more difficult for younger children, because many of them cannot easily carry out counting using this bundling process. A child might bundle 5 cookies, then 5 more, and so on until all cookies are gone, and then count the number of bundles. If so, this child is probably not counting by bundles. Think of the bundles as shrink-wrapped unit packages.

It is relatively straightforward to check if a child can count by units. For example, many 5 and 6 year olds can count, quite far, by 2's. Yet if we say: "I counted to 24 by 2's. How many times did I count?" children might get stuck on this counting question.

Here is an example that illustrates the advantages of counting by units. Ask how many 19's there are in 95. Many children, even as old as year 10 students, will find the answer to this question as follows: 19, 19+19=38, 38+19 = 57, 57+19 = 76, 76+19 = 95, so the answer is 5. Some children, however, answer as follows: 19 is 1 less than 20 and 5 lots of 20 is 100 (a remembered fact), so 5 lots of 19 is 5 less than 100, which is 95. These latter children are counting by units of 20 and carrying the relationship between 19 and 20 with then as they count.

The counting technique that is most helpful when we look at division of fractions is counting by bundled units. If a child is not strong on counting by whole number units, practicing this will be of great help when they come to division of fractions.

Division by fractions

Measurement models

We model division of fractions on quotitive division – counting by bundled units.

What is of great use in explaining fraction division to children is a measurement model of fractions.

The usual model of a fraction using a whole number of equal parts of an object, such as a pie or a candy bar, is valid, and it is the model with which most teachers are very familiar - which is why they probably use it exclusively. However, modeling fractions as potentially continuous parts of a continuously divisible quantity, such as time, or volume – of water, for example – or space, is much more helpful to a child when learning fraction division. We call this a measurement model because we use an arbitrary part of an object with which to measure. We take this measurement unit and repeat it as many times as needed to get a quantity equal to what we are measuring.

Let's use time and water as our continuously variable quantity in the following examples. You could use length - for example, length of a snowboard track – or even a pie if you allow arbitrary divisions of the pie, and not just divisions into whole numbers of pieces.

Measurement models are very useful for helping children make a transition to fractions as numbers in whole number division problems.

For example, children are commonly taught in the early years that the result of diving 2 into 7 is 3, remainder 1. This type of answer does not focus on the relationship of the remainder to the unit of division.

Suppose a child is asked to distribute 7 cookies evenly to 2 toy dolls. The child shares out 3 cookies to each doll and has 1 cookie left over – a remainder of 1 cookie. If we want all the cookies shared out to the two dolls, and if the cookies are able to be split into pieces, then each doll gets $3\frac{1}{2}$ cookies. So the result of the division of 2 into 7 is not 3 remainder 1, but $3\frac{1}{2}$.

Fractions come into play as numbers in a way that expresses a reality for the child. The fraction measures something that whole numbers cannot.

This becomes very clear if we use quantities such as liquid measure or time.

Suppose we want to share all of 7 cups of lemonade evenly between 3 children. We can give each child 2 cups of lemonade and then evenly share out $\frac{1}{3}$ of a cup more to each child. So the result of dividing 7 cups of lemonade by 3 is $2\frac{1}{3}$.

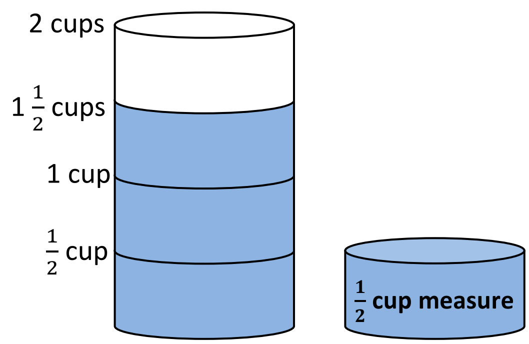
By using measurement models, teachers can help children make a transition to seeing fractions as numerical answers to sharing or division problems.

Simple examples of fraction division

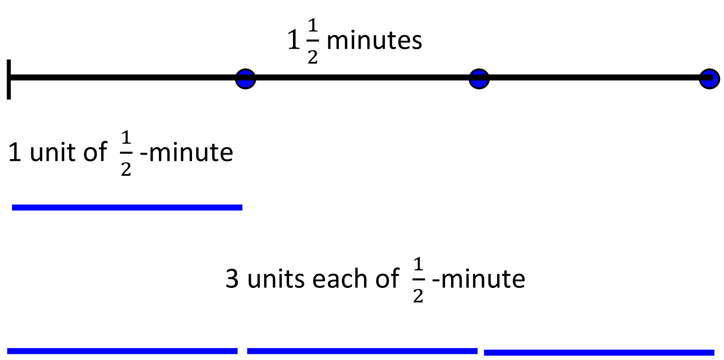
A simple question: what is $1\frac{1}{2} \div \frac{1}{2}$?

We interpret this as a counting question: "How many units of $\frac{1}{2}$ are there in $1\frac{1}{2}$?" or, to make it more concrete in terms of a water model, "How many **$\frac{1}{2}$-cups** are there in $1\frac{1}{2}$ cups ?"

Notice that, in this question, "**$\frac{1}{2}$-cup**" is the unit by which we measure. The answer, of course, is 3, as we can see from a diagram:



We can also interpret this question in terms of a time model: "How many **$\frac{1}{2}$-minutes** are there in $1\frac{1}{2}$ minutes?"



Notice, too, that the answer of "3" means 3 lots of **$\frac{1}{2}$-minutes**.

When we carry out division of fractions with children we need to keep track of the original quantity – $1\frac{1}{2}$ minutes in this example – and the measurement unit by which we are dividing – **$\frac{1}{2}$-minutes** in this example.

The result of the division – the answer – is always a certain number of units by which we are dividing – 3 **$\frac{1}{2}$-minutes** in this example.

Some more examples

1. How many $\frac{1}{3}$ minutes are there in 2 minutes?

2. What answer do you get when you divide $\frac{1}{4}$ of a minute into 2 minutes?

3. What is $1\frac{3}{4} \div \frac{1}{4}$?

Fraction division problems with fractional answers

The answer to a fraction division problem might not be a whole number, as it was in the examples above. It might be a fraction too.

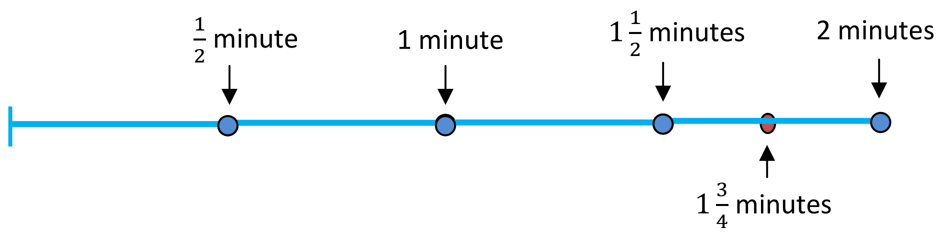
For example, let's ask: "How many **$\frac{1}{2}$-minutes** are there in $1\frac{3}{4}$ minutes ?"

We know that 3 **($\frac{1}{2}$-minutes)** is $1\frac{1}{2}$ minutes.

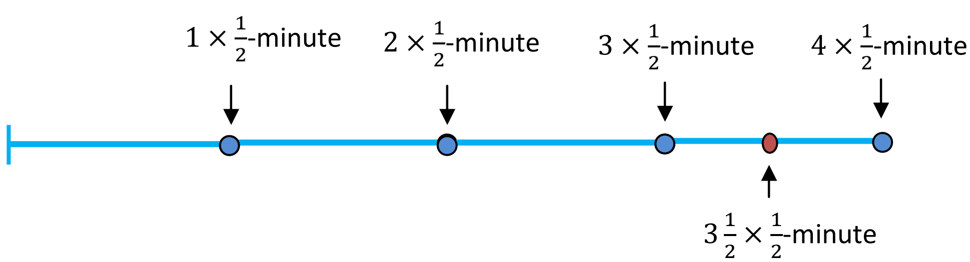
We can also see that 4 **($\frac{1}{2}$-minutes)** is 2 minutes.

So the answer to the question: "How many **$\frac{1}{2}$-minutes** are there in $1\frac{3}{4}$ minutes ?" is somewhere between 3 and 4.

How much more of a **$\frac{1}{2}$-minute unit** is there left in $1\frac{3}{4}$ minutes when we get to $1\frac{1}{2}$ minutes?



There is $\frac{1}{4}$ minute left over, and this is exactly $\frac{1}{2}$ of a **$\frac{1}{2}$-minute unit**.



So, there are exactly $3\frac{1}{2}$ (**$\frac{1}{2}$-minute) units** in $1\frac{3}{4}$ minutes.

This means, $1\frac{3}{4} \div \frac{1}{2} = 3\frac{1}{2}$

Common multiples

The question: "How many $\frac{1}{2}$-minutes are there in $1\frac{3}{4}$ minutes ?" can be looked at from a different perspective.

For two fractions, such as $\frac{1}{2}$ and $1\frac{3}{4}$, we ask the question: "What whole number multiple of both fractions will give a whole number answer in both cases?"

For example, $4 \times \frac{1}{2} = 2$ and $4 \times 1\frac{3}{4}$=7. The common multiple of 4, stretches the unit $\frac{1}{2}$-minute to 2 minutes, and stretches the quantity $1\frac{3}{4}$ minutes to 7 minutes.

Now, the number of $\frac{1}{2}$-minute units in $1\frac{3}{4}$ minutes is the same as the number of 2-minute units in 7 minutes, namely $\frac{7}{2} = 3\frac{1}{2}$. In other words $1\frac{3}{4} \div \frac{1}{2} = \frac{7}{2} = 3\frac{1}{2}$.

By stretching both the unit of a $\frac{1}{2}$-minute, and the quantity $1\frac{3}{4}$ minutes, by a common amount, we changed the fraction division problem $1\frac{3}{4} \div \frac{1}{2}$ into the whole number division problem $7 \div 2$ which has the fractional answer of $\frac{7}{2} = 3\frac{1}{2}$.

Some more examples

Figure out the answers to the following fraction division problems by finding common whole number multiples of the measurement unit, and of the quantity to be measured, that are also whole number quantities:

1. $1\frac{3}{4} \div \frac{1}{3}$

2. $2\frac{3}{5} \div \frac{1}{2}$

3. $3\frac{1}{4} \div \frac{1}{5}$

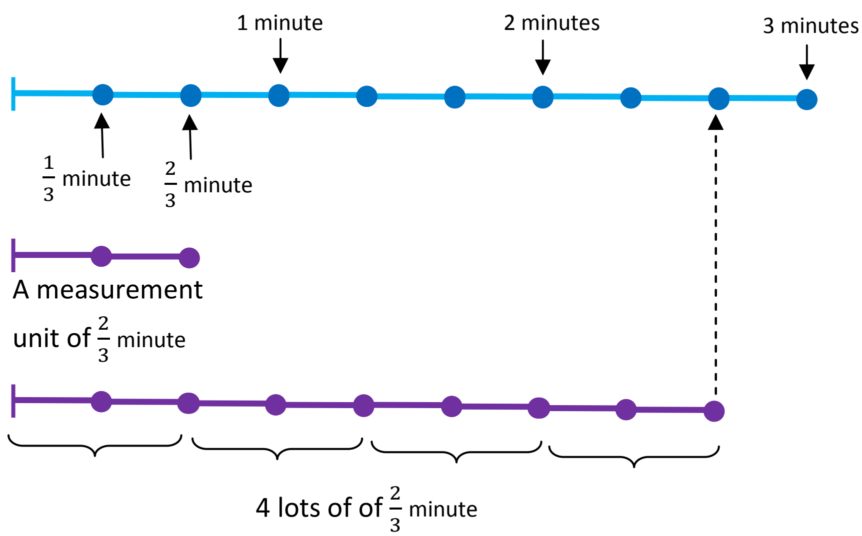
4. $1\frac{2}{7} \div \frac{2}{3}$

Keeping track of the unit of division

Notice how important it is to track, and keep a mental picture, or an actual drawing, of the unit by which we are dividing: the "$\frac{1}{2}$ " in the answer to $1\frac{3}{4} \div \frac{1}{2}$ means $\frac{1}{2}$ of a **$\frac{1}{2}$-minute unit.**

Let's illustrate this point again with the answer to the question: "How many $\frac{2}{3}$ –minutes are there in 3 minutes?"

We mark off a measurement unit of $\frac{2}{3}$ –minute and repeat it 4 times to get not quite 3 minutes. The bit left over is $\frac{1}{3}$ of a minute:



This bit left over is $\frac{1}{2}$ of the measurement unit of $\frac{2}{3}$ –minute.

So there are $4\frac{1}{2}$ units of $\frac{2}{3}$ –minute in 3 minutes. We write this as $3 \div \frac{2}{3} = 4\frac{1}{2}$.

To illustrate this point again, let's ask the question: "How many $\frac{2}{11}$ –minutes are there in 3 minutes?"

Does this look impossibly hard? Should we revert to "invert and multiply?" Not yet! Because we want to get the conceptual basis of division by fractions as counting by units, in place before we jump to unexplained procedures. Mathematics is not inexplicable magic – it is a rational, clear, logical process of thought, that – properly explained – can be understood by anyone.

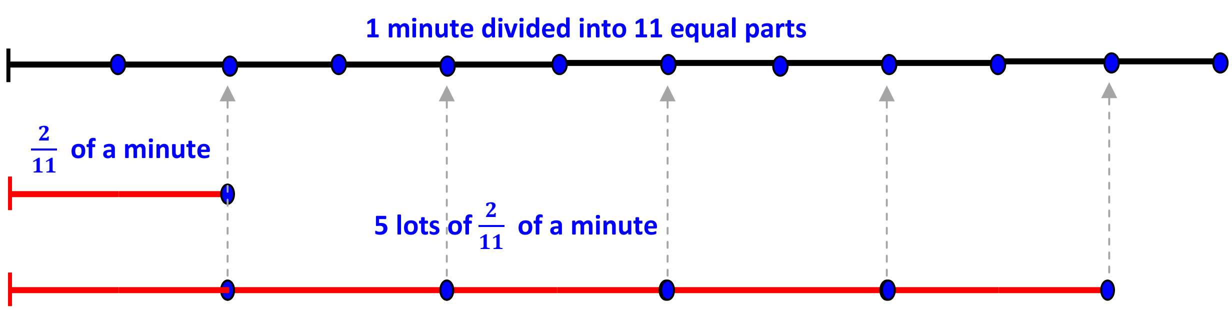
The thing to bear in mind with this question – as for the question: "How many **$\frac{1}{2}$-minutes** are there in $1\frac{3}{4}$ minutes?" – is the following:

> When a child first learns to count, the question: "How many?" is answered by a whole number. But not all "How many?" questions have a whole number answer: "How many 2's are there in 11" for example. Fractions help us answer "How many?" questions like the latter.

The fractional answer to "How many 2's are there in 11" is not "5 with remainder 1" but "$5\frac{1}{2}$". Thinking of 2 as the unit of measurement, the quantity 11 is $5\frac{1}{2}$ lots of 2.

To answer: "How many $\frac{2}{11}$ –minutes are there in 3 minutes?" – that is, to calculate $3 \div \frac{2}{11}$ – we will first try to answer "How many $\frac{2}{11}$ –minutes are there in 1 minute?".

Let's imagine 1 minute divided equally into 11 equal parts, and then let's count by units of $\frac{2}{11}$ –minutes:



We see that 5 lots of $\frac{2}{11}$ –minutes is $\frac{10}{11}$ of 1 minute.

To get to 1 minute we need $\frac{1}{11}$ of a minute more.

Now here is a really important bit: $\frac{1}{11}$ of a minute is $\frac{1}{2}$ of the unit we are using to measure – the unit we are dividing by – namely $\frac{2}{11}$ of a minute.

So there are $5\frac{1}{2}$ lots of $\frac{2}{11}$-minutes in 1 minute. In other words, using division notation: $1 \div \frac{2}{11} = 5\frac{1}{2}$.

To answer the original question: "How many $\frac{2}{11}$ –minutes are there in 3 minutes?" we multiply this answer by 3 to get $16\frac{1}{2}$ lots of $\frac{2}{11}$-minutes in 3 minutes. Using division notation: $3 \div \frac{2}{11} = 16\frac{1}{2}$.

Invert and multiply

A commonly stated rule for fraction division is "invert and multiply".

For example to carry out the fraction division $3 \div \frac{2}{11}$ we "invert" the quantity $\frac{2}{11}$ to get $\frac{11}{2}$ and then multiply: $3 \times \frac{11}{2} = \frac{33}{2} = 16\frac{1}{2}$.

This is the same answer we got before, and it seems quick and easy.

Where did this rule come from?

Look back at the drawing, above, that shows 1 minute divided by $\frac{2}{11}$-minutes.

The "11" in $\frac{2}{11}$ tells us in how many pieces the 1 minute should be evenly divided. It is called the denominator of the fraction.

The "2" in $\frac{2}{11}$ tells us how many of these 11 equal parts of 1 minute we should bundle together to form a measurement unit. It is called the numerator of the fraction.

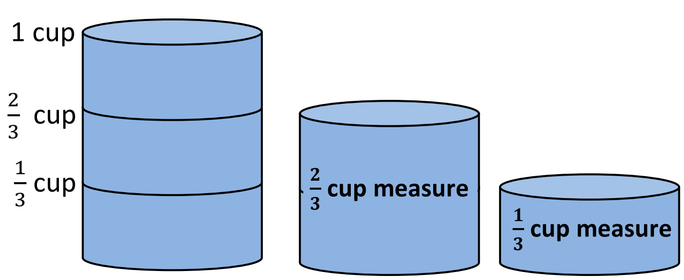
When we divide the 1 minute by $\frac{2}{11}$ we are counting how many of these $\frac{2}{11}$-units fit into 1 minute.

If we *invert* $\frac{2}{1}$ we get $\frac{1}{2}$, and $\frac{1}{2}$ of a $\frac{2}{11}$-unit is $\frac{1}{11}$.

There are exactly 11 of these $\frac{1}{11}$ units in 1 minute because we divided the minute into 11 equal time segments.

So there are $\frac{11}{2}$ lots of $\frac{2}{11}$-units in 1 minute. That is, $1 \div \frac{2}{11} = \frac{11}{2}$, which is the invert and multiply rule.

Here is another example. Suppose we want to figure out how many $\frac{2}{3}$ of a cup of water there are in 1 cup of water. That is we want to calculate $1 \div \frac{2}{3}$.



The answer, as we can see from the diagram, is $1\frac{1}{2}$ lots of $\frac{2}{3}$ cup measure.

If we *invert* $\frac{2}{1}$ we get $\frac{1}{2}$, and $\frac{1}{2}$ of a $\frac{2}{3}$-cup is $\frac{1}{3}$-cup.

There are exactly 3 of these $\frac{1}{3}$ - cups in 1 cup because we divided the cup into 3 equal parts.

So there are $\frac{3}{2}$ lots of $\frac{2}{3}$-cup in 1 cup, which again is the invert and multiply rule.

In the invert and multiply rule for this example, the numerator 2 is the count of how many multiples of $\frac{1}{3}$ make up the measurement unit of $\frac{2}{3}$. It becomes the denominator of the fraction of how many lots of $\frac{2}{3}$ we need to make 1 cup. The denominator 3 tells us how many of the $\frac{1}{3}$ units there are in 1 cup. It becomes the numerator of the fraction of how many lots of $\frac{2}{3}$ we need to make 1 cup.

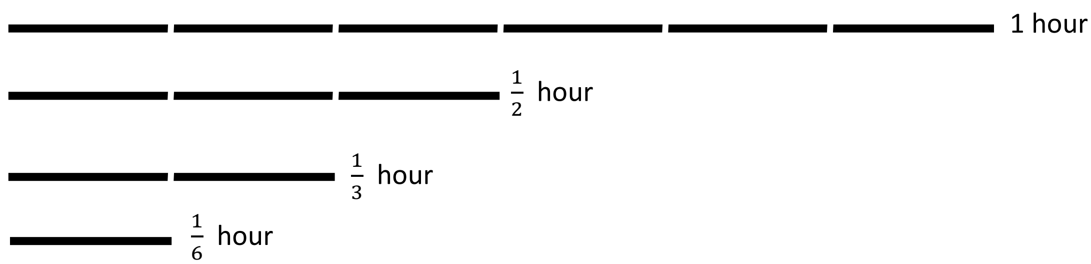
Some more examples

Explain the invert and multiply rule for the following fractions:

1. $\frac{1}{2}$	2. $\frac{1}{4}$	3. $\frac{1}{3}$
4. $\frac{2}{3}$	5. $\frac{3}{4}$	6. $\frac{3}{2}$

Common denominators

For some fraction division problems the unit of measurement with which are dividing, is not easily comparable to the quantity into which we are dividing. For example suppose we want to know how many $\frac{1}{3}$ hours are there in a $\frac{1}{2}$-hour. The quantity $\frac{1}{2}$ is not easily measurable directly by the unit $\frac{1}{3}$. In order to carry out this measurement, we can express both the quantity $\frac{1}{2}$ and the unit of measurement $\frac{1}{3}$ as multiples of a smaller unit of measurement. In this example we can choose $\frac{1}{6}$ as a measurement unit because then $\frac{1}{3} = 2 \times \frac{1}{6}$ and $\frac{1}{2} = 3 \times \frac{1}{6}$:



Looking at the diagram we can see that in the quantity $\frac{1}{2}$-hour, there are $1\frac{1}{2}$ lots of $\frac{1}{3}$ hours.

Some more examples

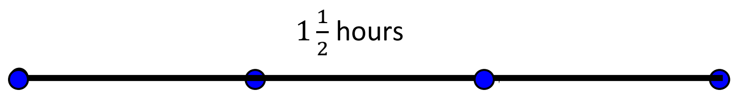
1. How many $\frac{1}{5}$ hours are there in a $\frac{1}{2}$ hour?

2. How many $\frac{2}{3}$ hours are there in $\frac{3}{4}$ hour?

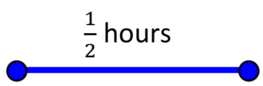
Measuring by units larger than the quantity to be measured

Fractions are wonderful in their flexibility! We can take a unit of measurement, such as $1\frac{1}{2}$ hours, and ask: "How many $1\frac{1}{2}$ hours are there in a $\frac{1}{2}$ hour?"

How many of these measurement units:



are there in this quantity:



Now, we cannot physically repeat the measurement unit $1\frac{1}{2}$ hours so many times to get the quantity $\frac{1}{2}$ hour, simply because the unit $1\frac{1}{2}$ hours is *bigger* than the quantity $\frac{1}{2}$ hour that we are trying to measure.

But we can measure the quantity $1\frac{1}{2}$ hours by the unit $\frac{1}{2}$-hour, and the result is 3: $1\frac{1}{2} \div \frac{1}{2} = 3$.

This means that 3 <u>equal</u> lots of $\frac{1}{2}$-hour is the same time as $1\frac{1}{2}$ hours. Expressed another way, this means a $\frac{1}{2}$-hour is $\frac{1}{3}$ of $1\frac{1}{2}$ hours.

So measuring a $\frac{1}{2}$-hour by units of $1\frac{1}{2}$ hour gives us $\frac{1}{3}$.

The quantity $\frac{1}{2}$ hour is $\frac{1}{3}$ of the measurement unit of $1\frac{1}{2}$ hours.

This example illustrates how important is the relationship between the measurement unit and the quantity to be measured. When the quantity to be measured is larger than the measurement unit, the result of the division is a whole number or fraction bigger than 1.

When the quantity to be measured is smaller than the measurement unit, the result of the division is a fraction smaller than 1.

Some more examples

1. How many $1\frac{2}{3}$ hours are there in a $\frac{1}{2}$ hour?

2. How many $2\frac{1}{4}$ hours are there in $\frac{3}{4}$ hour?

Fraction walls

Standard fraction walls

Fraction walls are in common use in elementary schools all over the world. Typically, a fraction wall consists of a large unit, denoted "1" and subdivisions of that unit into equal parts in various ways:

<table>
<tr><td colspan="30">1</td></tr>
<tr><td colspan="15">$\frac{1}{2}$</td><td colspan="15">$\frac{1}{2}$</td></tr>
<tr><td colspan="10">$\frac{1}{3}$</td><td colspan="10">$\frac{1}{3}$</td><td colspan="10">$\frac{1}{3}$</td></tr>
<tr><td colspan="6">$\frac{1}{5}$</td><td colspan="6">$\frac{1}{5}$</td><td colspan="6">$\frac{1}{5}$</td><td colspan="6">$\frac{1}{5}$</td><td colspan="6">$\frac{1}{5}$</td></tr>
<tr><td colspan="5">$\frac{1}{6}$</td><td colspan="5">$\frac{1}{6}$</td><td colspan="5">$\frac{1}{6}$</td><td colspan="5">$\frac{1}{6}$</td><td colspan="5">$\frac{1}{6}$</td><td colspan="5">$\frac{1}{6}$</td></tr>
</table>

The fraction wall above was constructed by making a table with 5 rows and 30 columns.

There were 30 columns because we need to have a multiple of $2 \times 3 \times 5 = 30$.

An easy way to do this is to construct the table in a spreadsheet and import the table into a word processing document.

The columns are then set to equal widths.

The columns in the first row were merged into one column, for the unit "1".

The columns in the second row were merged into two equal lots for the unit $\frac{1}{2}$ and so on for the remaining rows.

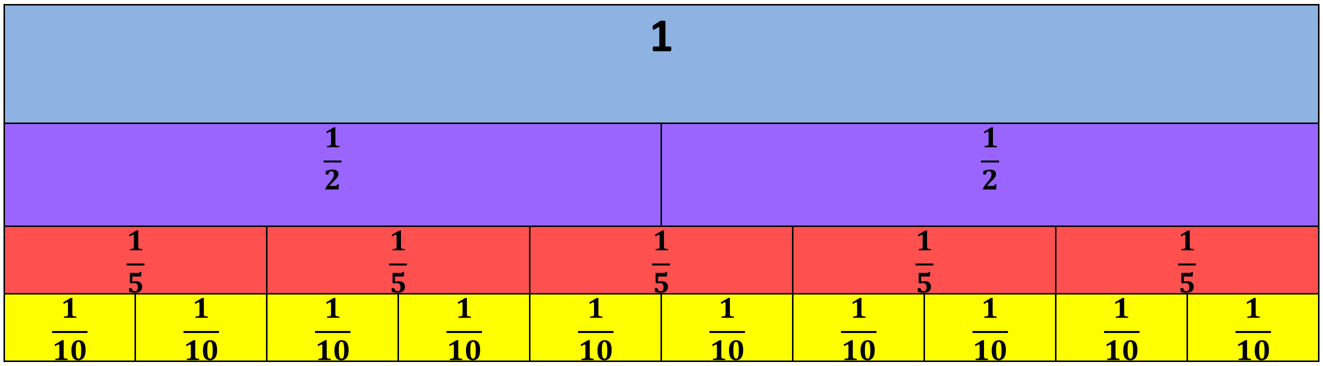
The reason that $\frac{1}{4}$ is not represented is that we would have needed 60 columns ($2 \times 2 \times 3 \times 5 = 60$) in order to have 4 lots of $\frac{1}{4}$, as well as equal lots of the other unit fractions.

The very important issue for a student of ***how*** we make equal parts of a continuous whole is done automatically by the word processor, as a hidden procedure. We do not know ***how*** it does this, just ***that*** it does.

We can also begin with a single column to represent "1" and use the split cells feature of the word processor to split cells into appropriate numbers of pieces.

How can we use the fraction wall to illustrate the division $\frac{1}{2} \div \frac{1}{5}$?

One way is to use a common denominator and measure both $\frac{1}{2}$ and division $\frac{1}{5}$ in terms of the smaller unit $\frac{1}{10}$:



The quantity $\frac{1}{2}$ is 5 lots of $\frac{1}{10}$, and the unit $\frac{1}{5}$ is 2 lots of the smaller unit $\frac{1}{10}$.

So there are $\frac{5}{2} = 2\frac{1}{2}$ lots of $\frac{1}{5}$ in $\frac{1}{2}$.

We can still use a measurement model by taking the “1” to represent, for example, 1 hour, or 1 mile, or 1 kilometer. With units of miles, for example, we are trying to measure how many $\frac{1}{5}$-mile units there are in $\frac{1}{2}$ a mile.

Fraction walls with fractional quantities greater than 1

Sometimes, as we have seen, we might want to divide a fraction larger than 1 by a fractional unit measure. For example we might want to figure out how many $\frac{1}{2}$- minutes there are in $1\frac{2}{3}$- minutes.

To illustrate this division with a fraction wall we can represent $1\frac{2}{3}$- minutes as an equal part of a larger whole number.

What whole number would that be, and what part of that whole number would $1\frac{2}{3}$- minutes be?

3 lots of $1\frac{2}{3}$- minutes is 5 minutes, so we could begin by dividing a strip representing 5 minutes into 3 equal pieces:

<table>
<tr><td colspan="15">5 minutes</td></tr>
<tr><td colspan="5">$1\frac{2}{3}$-minutes</td><td colspan="5">$1\frac{2}{3}$-minutes</td><td colspan="5">$1\frac{2}{3}$-minutes</td></tr>
<tr><td colspan="3">1 minute</td><td colspan="3">1 minute</td><td colspan="3">1 minute</td><td colspan="3">1 minute</td><td colspan="3">1 minute</td></tr>
</table>

We can now show the $\frac{1}{2}$ - minute unit on the same fraction wall:

<table>
<tr><td colspan="30">5 minutes</td></tr>
<tr><td colspan="10">$1\frac{2}{3}$-minutes</td><td colspan="10">$1\frac{2}{3}$-minutes</td><td colspan="10">$1\frac{2}{3}$-minutes</td></tr>
<tr><td colspan="6">1 minute</td><td colspan="6">1 minute</td><td colspan="6">1 minute</td><td colspan="6">1 minute</td><td colspan="6">1 minute</td></tr>
<tr><td colspan="3">$\frac{1}{2}$- minute</td><td colspan="3">$\frac{1}{2}$- minute</td><td colspan="3">$\frac{1}{2}$- minute</td><td colspan="3">$\frac{1}{2}$- minute</td><td colspan="3">$\frac{1}{2}$- minute</td><td colspan="3">$\frac{1}{2}$- minute</td><td colspan="3">$\frac{1}{2}$- minute</td><td colspan="3">$\frac{1}{2}$- minute</td><td colspan="3">$\frac{1}{2}$- minute</td><td colspan="3">$\frac{1}{2}$- minute</td></tr>
</table>

The problem now is that the quantity $1\frac{2}{3}$-minutes and the measurement unit $\frac{1}{2}$ – minute do not evenly overlap, so, again, it helps to measure both in terms of the smaller unit of $\frac{1}{6}$ – minute:

<table>
<tr><td colspan="30">5 minutes</td></tr>
<tr><td colspan="10">$1\frac{2}{3}$-minutes</td><td colspan="10">$1\frac{2}{3}$-minutes</td><td colspan="10">$1\frac{2}{3}$-minutes</td></tr>
<tr><td colspan="6">1 minute</td><td colspan="6">1 minute</td><td colspan="6">1 minute</td><td colspan="6">1 minute</td><td colspan="6">1 minute</td></tr>
<tr><td colspan="3">$\frac{1}{2}$ - minute</td><td colspan="3">$\frac{1}{2}$ - minute</td><td colspan="3">$\frac{1}{2}$ - minute</td><td colspan="3">$\frac{1}{2}$ - minute</td><td colspan="3">$\frac{1}{2}$ - minute</td><td colspan="3">$\frac{1}{2}$ - minute</td><td colspan="3">$\frac{1}{2}$ - minute</td><td colspan="3">$\frac{1}{2}$ - minute</td><td colspan="3">$\frac{1}{2}$ - minute</td><td colspan="3">$\frac{1}{2}$ - minute</td></tr>
<tr><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td></tr>
</table>

Now we can see that quantity $1\frac{2}{3}$-minutes is 10 lots of the unit $\frac{1}{6}$ – minute.

Also, the $\frac{1}{2}$ – minute unit is 3 lots of the unit $\frac{1}{6}$ - minute.

So $1\frac{2}{3} \div \frac{1}{2} = \frac{10}{3} = 3\frac{1}{3}$.

The use of fraction walls gets more difficult as we have to split the fraction measurement unit into finer pieces to accommodate the fractional quantity being measured.

Division of a quantity into an equal whole number of parts

When using fraction walls we leave the equal division of a quantity into several equal parts to the word processing software: we know ***that*** it can do this, but neither we nor students know **how**.

Using word processing software's ability to reproduce parallel lines, we can carry out the task of division of a quantity into an equal whole number of pieces by the construction described below.

This construction could also be carried out in Geometer's Sketchpad or Cabri.

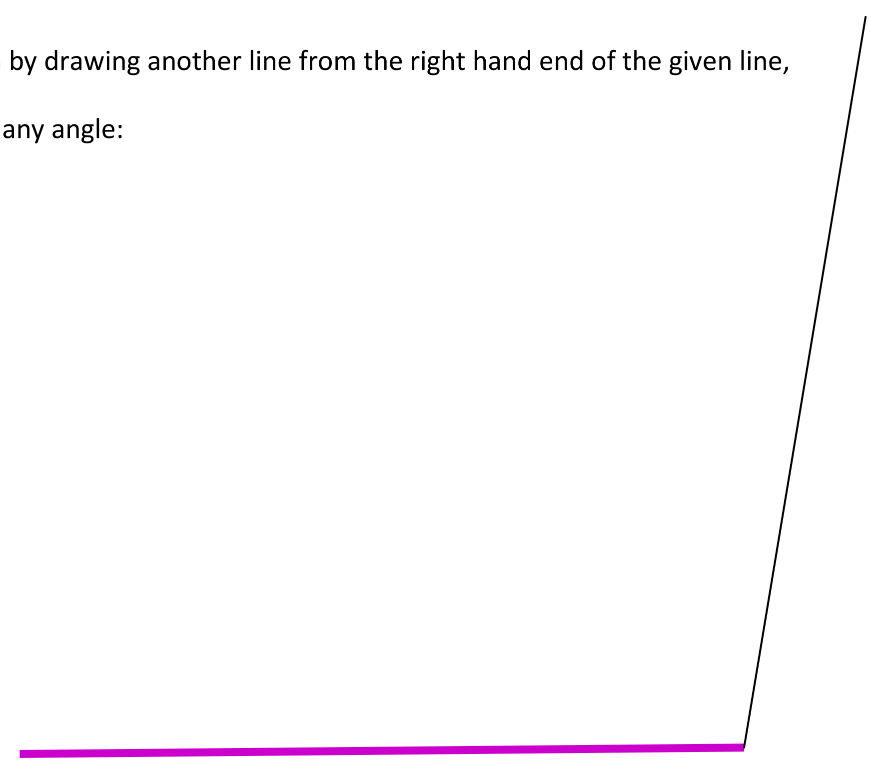
We will represent the quantity to be subdivided by a line:

Let's divide this line into 5 equal pieces.

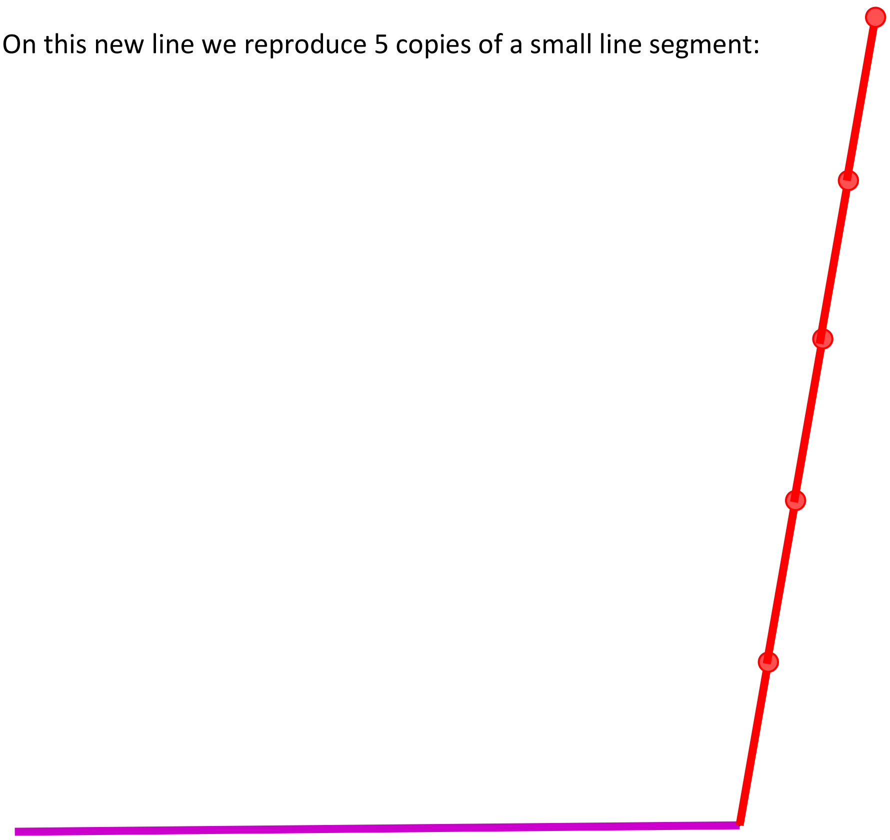
We choose "5" because paper folding will allow most people to divide a line, or sheet of paper, into 2 or 3 equal pieces.

We begin by drawing another line from the right hand end of the given line,

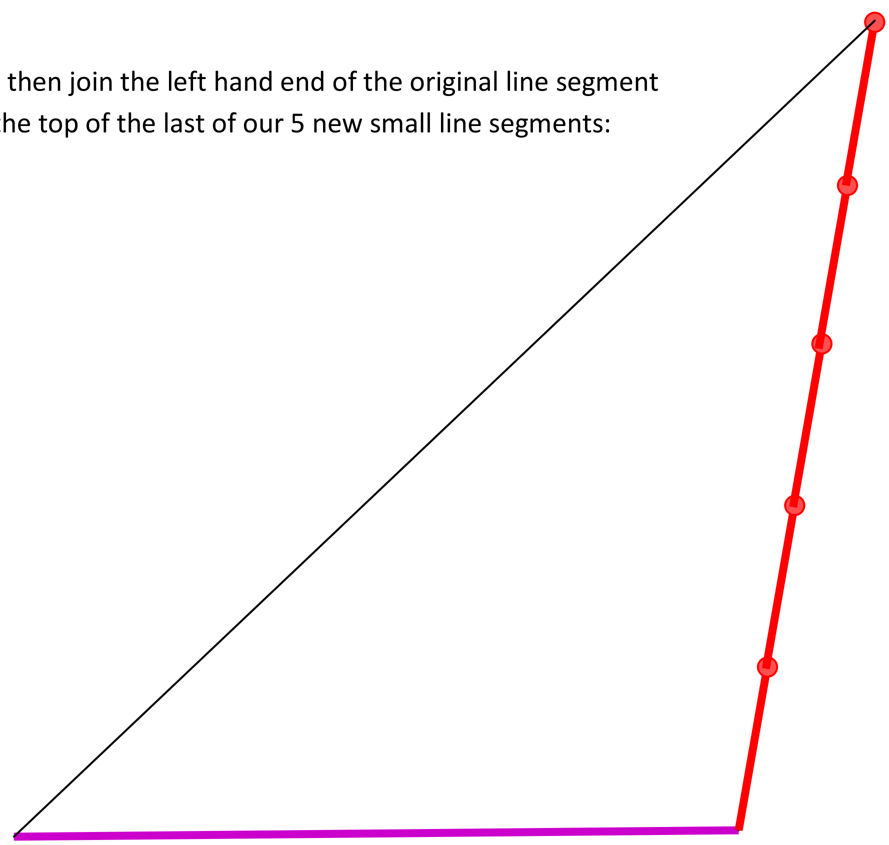
drawn at any angle:



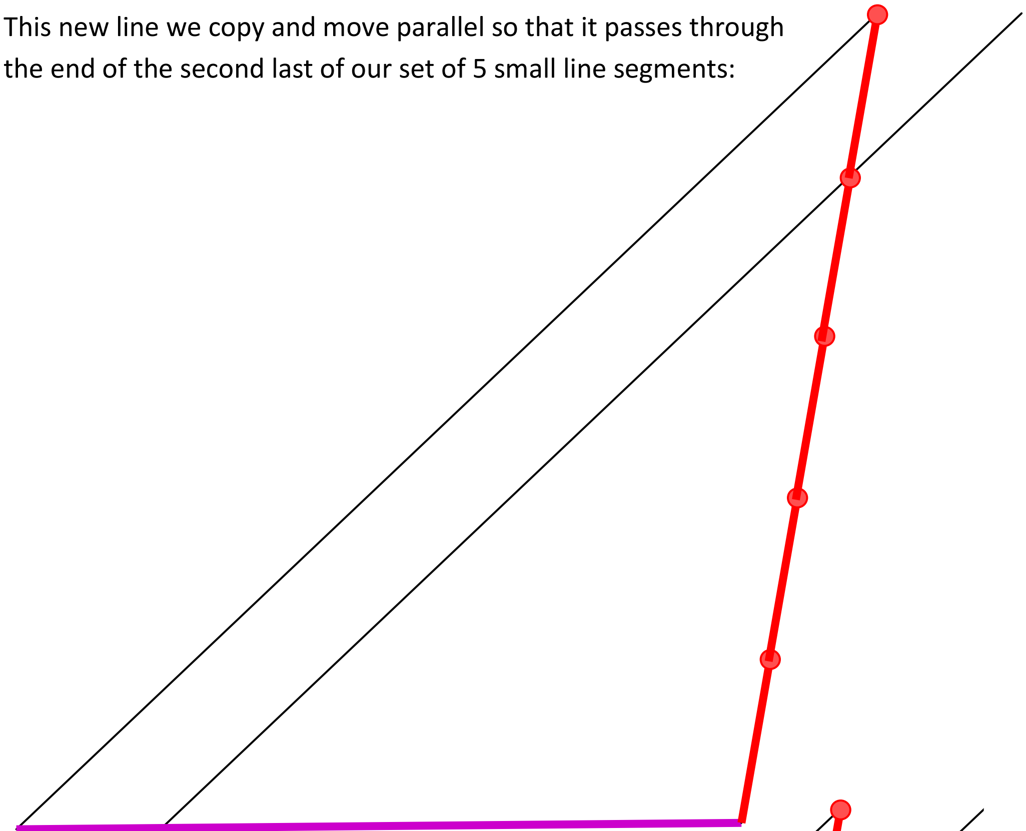
On this new line we reproduce 5 copies of a small line segment:



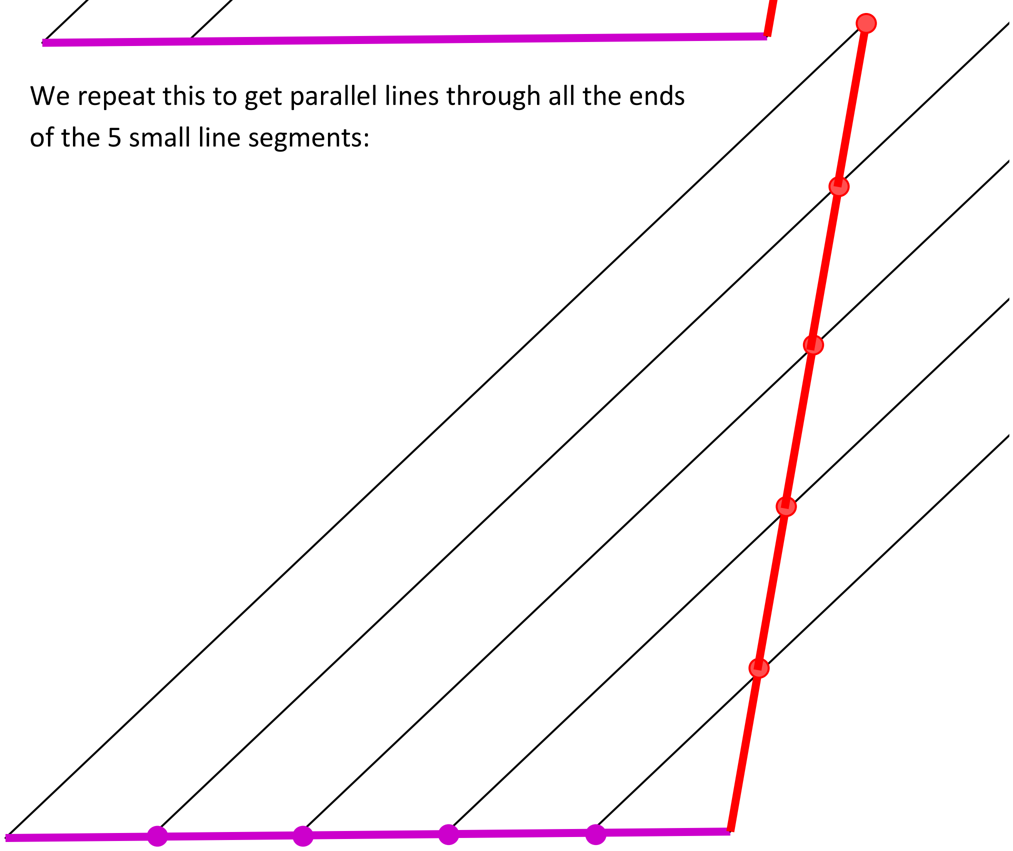
We then join the left hand end of the original line segment to the top of the last of our 5 new small line segments:



This new line we copy and move parallel so that it passes through the end of the second last of our set of 5 small line segments:



We repeat this to get parallel lines through all the ends of the 5 small line segments:

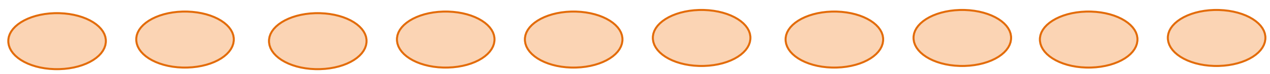


The points where these parallel lines meet the original line divide that line into 5 equal parts.

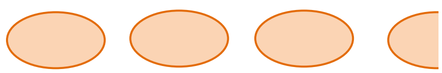
Worked word problems

Q 1. At a party there were 10 pizzas. The boys ate $6\frac{1}{2}$ pizzas, and the girls ate $3\frac{1}{2}$ pizzas. What fraction of the 10 pizzas did the girls eat, and what fraction did the boys eat?

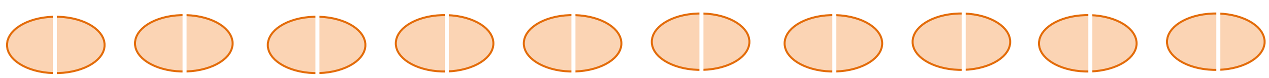
There were 10 pizzas:



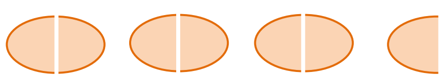
The girls ate $3\frac{1}{2}$ pizzas:



If we measure in units of $\frac{1}{2}$-pizza units, there were 20 lots of $\frac{1}{2}$-pizzas:



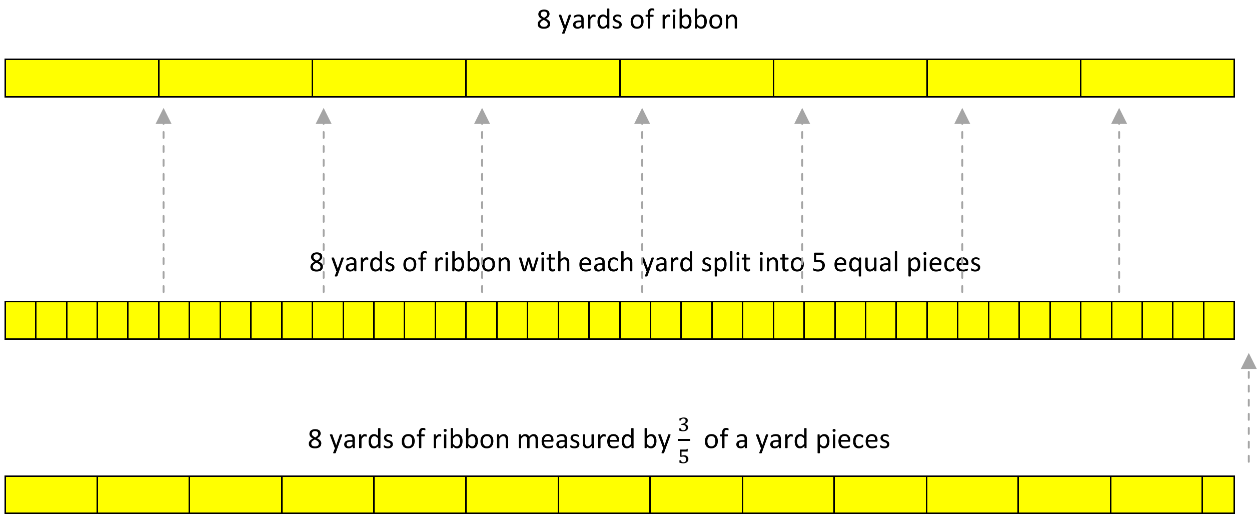
and the girls ate 7 of these $\frac{1}{2}$-pizzas:



So the girls ate $\frac{7}{20}$ of the pizzas.

Using the same units of $\frac{1}{2}$-pizza we see that the boys ate $\frac{13}{20}$ of the pizzas.

Q 2. Melissa was making ribbons for presents. Each ribbon was $\frac{3}{5}$ of a yard. The local craft store had 8 yards of ribbon. Melissa thought that she might not be able to make an exact while number of ribbons without a bit left over. She decided to use the bit left over to make a smaller ribbon for another present. How many whole ribbons could Melissa make from the ribbon in the store, and what fraction of a ribbon would be left over for the smaller ribbon?



There are 13 lots of $\frac{3}{5}$ of a yard, and $\frac{1}{5}$ of a yard left over.

The bit left over is $\frac{1}{3}$ of $\frac{3}{5}$ of a yard, so Melissa can make $13\frac{1}{3}$ ribbons: 13 big ribbons and 1 small ribbon that is $\frac{1}{3}$ the size of the larger ribbons.

Q 3. Bob pumped water from a tank to water his vegetable garden. He used $5\frac{1}{2}$ liters of water for every square meter of vegetable garden. The tank held 180 liters of water. How many square meters of vegetable patch could Bob water?

We have to figure out how many units of $5\frac{1}{2}$ liters there are in 180 liters: in other words, to calculate $180 \div 5\frac{1}{2}$.

One way to do this is to imagine we double the amount of water needed per square meter, and also double the amount of water in the tank.

Imagine we needed 11 liters for each square meter, and that there were 360 liters in the tank. Then we could water $\frac{360}{11} = 32\frac{8}{11}$ square meters.

So that is also the number of square meters we can water using a tank with 180 liters and using $5\frac{1}{2}$ liters of water for every square meter of vegetable garden.

Q 4. Shelley wanted to save to buy a present that cost 5 times her weekly allowance. She decided to save $\frac{2}{3}$ of her allowance each week. How many weeks would she need to save to be able to buy the present?

We need to work out how many units of $\frac{2}{3}$ there are in 5. In other words, we need to calculate $10 \div \frac{2}{3}$.

5 times Shelley's weekly allowance, with units of $\frac{1}{3}$ and $\frac{2}{3}$ of her allowance

<table>
<tr><td colspan="3">1</td><td colspan="3">1</td><td colspan="3">1</td><td colspan="3">1</td><td colspan="3">1</td></tr>
<tr><td>$\frac{1}{3}$</td><td>$\frac{1}{3}$</td><td>$\frac{1}{3}$</td><td>$\frac{1}{3}$</td><td>$\frac{1}{3}$</td><td>$\frac{1}{3}$</td><td>$\frac{1}{3}$</td><td>$\frac{1}{3}$</td><td>$\frac{1}{3}$</td><td>$\frac{1}{3}$</td><td>$\frac{1}{3}$</td><td>$\frac{1}{3}$</td><td>$\frac{1}{3}$</td><td>$\frac{1}{3}$</td><td>$\frac{1}{3}$</td></tr>
<tr><td colspan="2">$\frac{2}{3}$</td><td colspan="2">$\frac{2}{3}$</td><td colspan="2">$\frac{2}{3}$</td><td colspan="2">$\frac{2}{3}$</td><td colspan="2">$\frac{2}{3}$</td><td colspan="2">$\frac{2}{3}$</td><td colspan="2">$\frac{2}{3}$</td><td>$\frac{1}{3}$</td></tr>
</table>

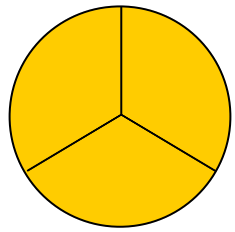
There are 7 lots of $\frac{2}{3}$ of Shelley's allowance in 5 times her allowance, plus a small bit of $\frac{1}{3}$ of her allowance left over.

That extra $\frac{1}{3}$ of her allowance is $\frac{1}{2}$ of the measurement unit of $\frac{2}{3}$ of her allowance.

So Shelley will have to save for $7\frac{1}{2}$ weeks.

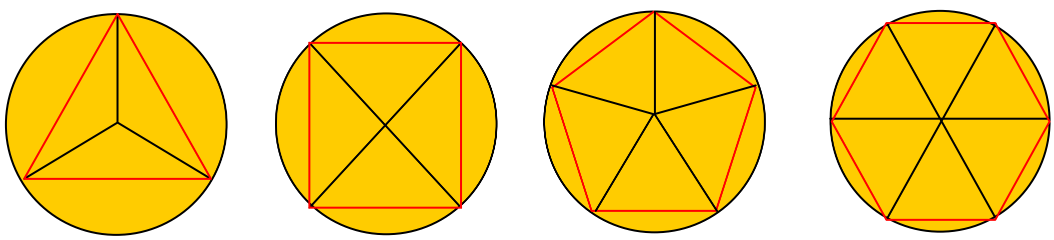
Why splitting pies into equal slices is not a good idea for teaching fractions

A common model for discussing fractions is spitting a pie into several equal pieces using radial cuts – straight line cuts from the center of the pie to the crust:



The problem with this is that when we make the first cut, it is not obvious, other than guessing, where we should make the second, and following, cuts.

If we can successfully make the cuts so as to divide the pie into equal slices then, by joining the points where the cuts meet the edge – the crust – we get a regular polygon:



The problem is that there is generally no good method for constructing regular polygons.

Regular polygons with 3, 4, 5, and 6 sides can be constructed using straight lines and circles, but a regular polygon with 7 sides cannot be constructed using straight lines and circles.

The question of which regular polygons can be constructed, and how, is fairly complicated:
http://en.wikipedia.org/wiki/Constructible_polygon

You may object that we can draw a regular polygon by measuring the internal angles with a protractor. However, that method relies on setting up degrees on a protractor as an arbitrary unit of measure. Our objective in fraction division is to use one fractional part of an object to measure another, and placing degree measure center stage avoids the issue. This would be like solving fraction problems by measuring lengths in millimeters, or time in seconds. It avoids the fraction division by going to an arbitrary fine unit of measurement.

Because equal division of a pie by radial slices gives a regular polygon by joining the ends of the slices, the construction of equal pieces of a pie is just as hard as constructing a regular polygon. This is a hard problem.

In the interests of truth, honesty, transparency, and full disclosure, we should tell our students this, and not pretend that we know where to make the cuts in a pie so as to get equal pieces when we are talking about fractions.

If, in a practical application, we did have to divide a pie *very* carefully, then using a protractor would be a good method. However, realistically, how often do we need to be that careful? Mostly we will guess. But we should tell our students we are guessing.

The authors

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Gary Davis has over 35 years experience in successfully teaching mathematics, and more than 20 years experience in researching how children succeed in learning mathematics. He has years of relevant experience in how children succeed at mathematics in the USA, UK, and Australia. With Robert Hunting he is co-author of Early Fraction Learning, published by Springer-Verlag. He has been the Boeing Distinguished Professor of Mathematics Education at Washington State University, Visiting Professor of Mathematics Education at Rutgers University in New Jersey, Visiting Professor in the Mathematics Education Research Centre, University of Warwick, UK, Professor of Education at the University of Southampton, UK, and Director of the Institute for Mathematics Education, LaTrobe University, Melbourne, Australia. He has lectured and given presentations on his work throughout the world.

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Cath Pearn is an experienced elementary teacher with a Master's degree in mathematics education. She has been Education Officer responsible for Numeracy and Mathematics at the Catholic Education Office, Melbourne, Australia, formerly a Research Fellow at the Australian Council for Educational Research (ACER), and a mathematics education lecturer for pre-service elementary teachers at LaTrobe University in Melbourne. She was an invited member for the Australian Federal Government's Numeracy Benchmarks, a member of the Mathematics Education Research Group of Australasia, the Association of Teacher Educators of Europe, and was a councilor for the Mathematics Association of Victoria for 5 years. Cath is currently completing her PhD on the role of language in young children's mathematical development. Working for ACER she has developed a Mathematics Intervention program for Grade 1 students at risk in mathematics and a similar program for Grades 3 and 4 students. She has lectured and given presentations on her work in Australia, the Czech Republic, Ireland, Spain, the UK, and the United States